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OPTIMUM ENERGY FLIGHTS TAKING INTO ACCOUNT THE EFFECT OF LIGHT PRESSURE *

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SUMMARY

Consideration is given in the present work to one-impulse flights between Keplerian orbits of given boundary in the gravitational field of the Sun. All of the three orbits are coplanar. The initial mass is minimized. The pressure of sunlight is taken into account on the intermediate orbit of the flight. Analytical solutions are obtained in the special cases of circular boundary orbits and on boundaries of orbits with small eccentricities. author

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We shall consider the question of constructing one-impulse trajectories of interorbital flights, with minimum mass consumption, in the Sun's gravitational field of a spherically-symmetrical, light-emitting central body. The perturbing effect upon the cosmic object by the celestial objects situated on the given boundary Keplerian orbits will be neglected. It is assumed that all three orbits — the initial, intermediate and final — lie in the same plane and that the motion along them takes place in a single direction.

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* ENERGETICHESKI OPTIMAL'NYYE POLETY S UCHETOM VLIYANIYA SVETOVOGO DAVLENIYA.

On the one hand, the pressure of light may be considered as a small corrective factor, namely in the course of flights by standard spaceships. On the other hand, in case of flights of interplanetary probes, consisting of thin hollow shells — balloons, filled with low-pressure gas and covered by an outside coating, well reflecting the light, it becomes comparable in magnitude with the force of attraction of the Sun. Ehricke [1] proposed to utilize such shells for the investigation of space of our solar system, and also for the transfer of useful payloads. These probes are characterized by their simplicity, small weight and capability of carrying a payload. Such objects are very bright, thereby increasing the probability of their successful tracking by telescopes. The shell-sondes may be launched from the ground, as well as from spaceships or artificial satellites.

1. - LIGHT PRESSURE

Let us find the principal vector and the principle moment of light pressure forces acting upon a fixed body with irradiated surface S. In deriving the formulas in this part, we shall start from the quantum theory of light and we shall neglect the effect conditioned by the non-isotropy of over-irradiation.

We shall introduce a fixed system of coordinates x, y, z; the axis z coincides with the heliocentric radius-vector of the irradiated body, that is, it is directed along the parallel light beam incident upon it. Let us outline on body surface the elementary area (dS), R being its reflection factor. Assume $\bar{\rho} = \{\xi \; \eta \; \zeta\}$ as being the radius-vector of any point on (dS), the origin of this vector being at the point O of the irradiated body; α is the angle of incidence, equal to reflection angle; β is the angle between the projection of the external normal to the area on the plane xyz and the axis x, counted from the axis x in the positive direction. The elementary variation of the quantity of motion for the time dt will then be

$$d\bar{K} = (M_2 c\bar{e}_2 - M_1 c\bar{e}_1)\,dt, \qquad (1)$$

where c is the speed of light; $\bar{e}_1$ is the unitary ort of the incident beam; $\overline{e}_2$ is the unitary ort of the direction of the reflected beam of rays; M_1 and M_2 are respectively the masses of photons, incident upon the area (dS) and reflected from it per unit of time.

Starting from the principle of mass and energy equivalence, we shall find

$$M_1 = \frac{E}{c^2}\cos\alpha\, dS, \quad M_2 = \mathrm{R}\frac{E}{c^2}\cos\alpha\, dS. \tag{2}$$

Here E is the solar constant (power of solar radiation corresponding to area unit) for the area situated at the distance r from the Sun; it can be computed as follows :

$$E = \frac{E_{\delta} r_{\delta}^2}{r^2}, \quad E_{\delta} r_{\delta}^2 = 0.302\cdot 10^{33} \text{ erg/sec} \tag{3}$$

where r_{δ} is the average distance between Earth and Sun, and E_{δ} is the solar constant for the distance r_{δ}. Utilizing the theorem for the quantity of motion, we shall find the force $\overline{f}$ of light pressure upon the area (dS)

$$\left.\begin{aligned} f_x &= -\mathrm{R}\frac{E}{c}\sin 2\alpha\cos\alpha\cos\beta\, dS,\\ f_y &= -\mathrm{R}\frac{E}{c}\sin 2\alpha\cos\alpha\sin\beta\, dS,\\ f_z &= \frac{E}{c}(1+\mathrm{R}\cos 2\alpha)\cos\alpha\, dS. \end{aligned}\right\} \tag{4}$$

After that it is easy to find the projections of the principal vector $\overline{F}$ and of the principal moment $\overline{L}$ relative to the point O on the axis of the coordinates

$$\left.\begin{aligned} F_x &= -\frac{2E}{c}\iint\limits_{(S)}\mathrm{R}\cos^2\alpha\sin\alpha\cos\beta\, dS,\\ F_y &= -\frac{2E}{c}\iint\limits_{(S)}\mathrm{R}\cos^2\alpha\sin\alpha\sin\beta\, dS,\\ F_z &= \frac{E}{c}\iint\limits_{(S)}(1+\mathrm{R}\cos 2\alpha)\cos\alpha\, dS, \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} L_x &= \frac{E}{c}\iint\limits_{(S)}[\eta(1+\mathrm{R}\cos 2\alpha)+\zeta\mathrm{R}\sin 2\alpha\sin\beta]\cos\alpha\, dS,\\ L_y &= -\frac{E}{c}\iint\limits_{(S)}[\zeta\mathrm{R}\sin 2\alpha\cos\beta+\xi(1+\mathrm{R}\cos 2\alpha)]\cos\alpha\, dS,\\ L_z &= \frac{E}{c}\iint\limits_{(S)}\mathrm{R}(\eta\cos\beta-\xi\sin\beta)\sin 2\alpha\cos\alpha\, dS. \end{aligned}\right\} \tag{6}$$

We shall find the light pressure upon a spherical body of radius R with a reflection factor R, identical for the entire surface. Let us introduce on body surface a spherical system of coordinates with origin at the center of the sphere (ψ is the latitude, φ is the longitude); besides, we shall count the longitude φ from the direction at the Sun. Then,

$$\cos\alpha = \cos\varphi\cos\psi. \tag{7}$$

Because of symmetry, we shall reduce the system of light pressure forces to the resultant force, applied at the geometrical center of the sphere and equal to

$$F_z = \frac{E_\delta r_\delta^2}{cr^2}\pi R^2. \tag{8}$$

Therefore, the force acting upon a fixed irradiated spherical body is directed along its heliocentrical radius-vector and is not dependent on the latter's reflecting power — result quite analogous to the result obtained by Radziyevskiy [2]. Thus, a well reflecting coating should only be applied to prevent excessive heating.

Remark 1. - If the irradiated body moves with a velocity U, both, the magnitude of light pressure force and its direction vary by a quantity of the order $\frac{U}{c}$ (for more details, see, for example, [2]). Because of the smallness of the last ratio, we shall neglect these variations.

2. - HELIOCENTRICAL TRAJECTORIES TAKING INTO ACCOUNT THE LIGHT PRESSURE

Let us consider a body of mass m, moving in the fields of Newtonian gravitation and of Sun's light radiation, the mass of which we shall denote by M. The components F_x , F_y of light pressure forces will be neglected. These components are exactly zero for spherical bodies.

The force of attraction toward the Sun is $\frac{k^2Mm}{r^2}$, where k^2 is the gravitational constant, while the force of light repulsion is $\frac{B}{r^2}$, where for a sphere of radius R

$$B = \frac{E_\delta r_\delta^2}{c} \pi R^2, \tag{9}$$

while for other bodies, B is easy to find utilizing formulas (3) and (5). Writing the equation of relative motion of the irradiated body and assuming that $m \ll M$, we shall find for its heliocentrical radius-vector $\bar{r}$

$$\ddot{\bar{r}} = -\frac{k^2 M}{r^3}\bar{r} + \frac{B}{mr^3}\bar{r}. \tag{10}$$

Let us introduce the "reduced" mass of the Sun

$$M' = M(1-\delta), \tag{11}$$

where δ is a parameter characterizing the "decrease" of Sun's mass; it is equal to

$$\delta = \frac{B}{k^2 m M}. \tag{12}$$

Therefore, with a "reduced" mass of the Sun all formulas of the problem of two bodies will be valid. However, the quantity

$$K = k\sqrt{M}, \tag{13}$$

should be formally substituted everywhere by a new one :

$$\widetilde{K} = K\sqrt{1-\delta}. \tag{14}$$

We shall assume, that the parameter $\delta \in [0, 1)$. At $\delta = 1$, the gravitational attraction will be equilibrated by luminous repulsion, and rectilinear inertial flights are possible in any direction. At $\delta > 1$, the irradiated bodies will fly out of the solar system along hyperbolae, in which the Sun is located in the external focus.

Let us estimate the quantity δ for thin hollow spherical shell-sondes of radius R, with shell's thickness h and density γ. Assume that the shell-sonde carries a payload of mass m_0; then the total mass of the sonde will be

$$m = m_0 + 4\pi R^2 \gamma h \tag{15}$$

and the parameter

$$\delta = \frac{E_\delta r_\delta^2 \pi R^2}{K^2 c(m_0 + 4\pi R^2 \gamma h)} = \frac{E_\delta r_\delta^2}{K^2 c\left(\frac{m_0}{\pi R^2} + 4\gamma h\right)}. \tag{16}$$

If $m_0 = 0$, δ does not depend on the radius of the sonde, but only on the product γh. We shall compute a series of values for δ at various h and at $\gamma = 1$ g/cm^3, in the assumption, that m_0 is substantially smaller than the mass of the shell itself :

δ	0.5	0.1	0.05	0.01	0.005	0.001
h, mk	0.38	1.9	3.8	19.0	38.0	190

This estimate is in qualitative agreement with that conducted by Ehricke [1].

The constant elements of orbits with "reduced" Sun's mass shall be called geometrical. We shall take for such elements

$$p = \frac{1}{\sqrt{l}}, \quad q = \frac{e}{\sqrt{l}}, \quad \omega, \quad T, \tag{17}$$

where l is the focal parameter; e is the eccentricity; ω – the angular distance of the pericenter and T is the time of passing through the pericenter. We shall take the polar angle ϑ for the independent variable determining the position in the orbit, so that the positive direction of the count coincide with the direction of motion.

The quantities related to the osculating orbit will be provided with the index "ock". Then, we shall have :

$$U_r = Kq_{\text{ock}} \sin(\vartheta - \omega_{\text{ock}}) = K\sqrt{1-\delta}\, q \sin(\vartheta - \omega), \tag{18}$$

$$U_\vartheta = K[p_{\text{ock}} + q_{\text{ock}} \cos(\vartheta - \omega_{\text{ock}})] = K\sqrt{1-\delta}\,[p + q\cos(\vartheta - \omega)], \tag{19}$$

$$\frac{1}{r} = p^2_{\text{ock}} + p_{\text{ock}} q_{\text{ock}} \cos(\vartheta - \omega_{\text{ock}}) = p^2 + pq\cos(\vartheta - \omega), \tag{20}$$

$$\psi = K(t_2 - t_1) = \int\limits_{\vartheta_1 - \omega_{\text{ock}}}^{\vartheta_2 - \omega_{\text{ock}}} \frac{dv}{p_{\text{ock}}(p_{\text{ock}} + q_{\text{ock}} \cos v)^2} =$$

$$= \frac{1}{\sqrt{1-\delta}} \int\limits_{\vartheta_1 - \omega}^{\vartheta_2 - \omega} \frac{dv}{p(p + q\cos v)^2}, \tag{21}$$

where t_1, t_2 are the moments of time responding to polar angles ϑ_1, ϑ_2 respectively; U_r, U_ϑ are the radial and the transverse velocity components. Note that only the obvious dependence of the sub-integral function on ϑ should be taken into account in the first integral. Both integrals in (21) are easily taken (for example, for the elliptical motion through the eccentrical anomalies : the osculating one for the first integral, the geometrical for the second). From the correlations (18) — (21) it is easy to find

$$p_{\text{оск}} = \frac{p}{\sqrt{1-\delta}}, \tag{22}$$

$$q_{\text{оск}} = \left\{ q^2(1-\delta) - 2pq\delta\cos(\vartheta-\omega) + \frac{p^2\delta^2}{1-\delta} \right\}^{\frac{1}{2}}, \tag{23}$$

$$\operatorname{tg}\omega_{\text{оск}} = \frac{q(1-\delta)\sin\omega - p\delta\sin\vartheta}{q(1-\delta)\cos\omega - p\delta\cos\vartheta}, \tag{24}$$

$$T_{\text{оск}} = T + \frac{1}{K\sqrt{1-\delta}} \int\limits_{0}^{\vartheta-\omega} \frac{dv}{p(p+q\cos v)^2} - \frac{1}{K} \int\limits_{0}^{\vartheta-\omega_{\text{оск}}} \frac{dv}{p_{\text{оск}}(p_{\text{оск}}+q_{\text{оск}}\cos v)^2}, \tag{25}$$

and at the same time, the signs of the numerator and denominator in formula (24) coincide with the signs of $\sin\omega_{ock}$ and $\cos\omega_{ock}$, respectively, while when computing the second integral in formula (25), the osculating elements should be considered as constant. Note that all the osculating elements are periodical functions of the angle ϑ with a period 2π (at elliptical motion in geometrical elements).

3. - SETTING UP THE PROBLEM OF OPTIMUM ENERGY FLIGHT TAKING INTO ACCOUNT THE INFLUENCE OF LIGHT PRESSURE. SYSTEM OF INDISPENSABLE CONDITIONS

It is required to materialize the flight of a cosmic device between pre-assigned boundary orbits with the aid of a single impulse - that over the initial orbit. The value of the characteristic velocity of this impulse is minimized, which assures a minimum fuel consumption. All the orbits are coplanar. The light pressure is taken into account only over the intermediate orbit. Depending upon the concrete physical problem, we may use of the cosmic device the above-described shell-sonde, or any other spaceship, for which light pressure must be taken into account.

Assume that the initial orbit has for elements p_1, q_1, ω_1, T_1, and the final one — p_2, q_2, ω_2, T_2. It is assumed that in heavenly objects the ratio of effective cross section to mass on these orbits is small, and that the light pressure has practically no effect on orbit elements.

A boost-impulse is applied at the time t_1, when the polar angle is ϑ_1, as a result of which the spaceship passes to flight's intermediate orbit. The latter intersect the final orbit at the time t_2, at polar angle ϑ_2, and the spaceship collides with the heavenly object situated on the final orbit.

At pre-assigned elements of boundary orbits it is necessary to determine the intermediate orbit of the flight, that is, to find its geometrical elements p, q, ω and also the angles ϑ_1, ϑ_2 and the moments of time t_1, t_2, in such a way, that the magnitude of the characteristic velocity of the initial impulse have a minimum. Upon determination of these unknowns we shall obtain for the moment of passing through the pericenter

$$T = t_i - \frac{1}{K\sqrt{1-\delta}} \int_0^{\vartheta_i - \omega} \frac{dv}{p(p+q\cos v)^2}, \quad i=1,\ 2, \tag{26}$$

and the osculating elements can be computed by the formulas (22) — (25).

The following conditions should be fulfilled:

$$\varphi_1 = p^2 + pq\cos(\vartheta_1 - \omega) - p_1^2 - p_1 q_1 \cos(\vartheta_1 - \omega_1) = 0, \tag{27}$$

$$\varphi_2 = p^2 + pq\cos(\vartheta_2 - \omega) - p_2^2 - p_2 q_2 \cos(\vartheta_2 - \omega_2) = 0, \tag{28}$$

$$\varphi_3 = \psi_1 + \psi - \psi_2 - \alpha = 0, \tag{29}$$

where the function ψ is expressed through the geometrical elements by the formula (21), but

$$\psi_i = K(t_i - T_i) = \int_0^{\vartheta_i - \omega_i} \frac{dv}{p_i(p_i + q_i\cos v)^2}, \quad i=1,\ 2, \tag{30}$$

$$\alpha = K(T_2 - T_1). \tag{31}$$

The correlations (27) and (28) imply the continuity of radii-vectors at starting and finishing points, while the correlation (29) is the condition of motion time coincidence prior to encounter at the final point along the initial and intermediate orbits on the one hand, and along the final — on the other.

The characteristic velocity ΔU of the initial impulse may be expressed through elements of the initial and intermediate orbits (see [3]) for analogous operations), as follows :

$$\Delta U = K\Big\{q^2(1-\delta) - p^2(1-\delta) + q_1^2 + 3p_1^2 - 2p_1^2\delta - \frac{2p_1^3\sqrt{1-\delta}}{p} -$$
$$- 2qq_1\sqrt{1-\delta}\cos(\omega_1-\omega) -$$
$$- 2q_1\sqrt{1-\delta}\Big[p + \frac{p_1^2}{p} - p_1\frac{2-\delta}{\sqrt{1-\delta}}\Big]\cos(\vartheta_1-\omega_1)\Big\}^{\frac{1}{2}}, \qquad (32)$$

and for the inclination angle of the thrust Φ (counted from the transversal in a direction opposite to that of the motion) we have

$$\operatorname{tg}\Phi = \frac{q\sqrt{1-\delta}\sin(\vartheta_1-\omega) - q_1\sin(\vartheta_1-\omega_1)}{\left(\frac{p_1\sqrt{1-\delta}}{p}-1\right)[p_1+q_1\cos(\vartheta_1-\omega_1)]}, \qquad (33)$$

the signs of the numerator and denominator coinciding with those of, respectively, $\sin\Phi$ and $\cos\Phi$.

We shall seek the minimum of the function

$$g = \frac{(\Delta U)^2}{2K^2\sqrt{1-\delta}} \qquad (34)$$

in the class of variables $p, q, \omega, \vartheta_1, \vartheta_2,$ which are dependent upon and linked by the conditions (27) – (29), that is, we find ourselves in the class of the conditional extremum of a function of finite number of variables. Introducing the constant multipliers $\lambda_1, \lambda_2, \lambda_3$, we shall compose the Lagrange function

$$g + \sum_{i=1}^{3}\lambda_i\varphi_i. \qquad (35)$$

As is well known, the partial derivatives of the Lagrange function by all variables must be zero

$$q_1p_1\Big[\frac{p}{p_1} + \frac{p_1}{p} - \frac{2-\delta}{\sqrt{1-\delta}}\Big]\sin(\vartheta_1-\omega_1) +$$
$$+ \lambda_1[p_1q_1\sin(\vartheta_1-\omega_1) - pq\sin(\vartheta_1-\omega)] + \lambda_3\Big(\frac{\partial\psi}{\partial\vartheta_1} + \frac{\partial\psi_1}{\partial\vartheta_1}\Big) = 0, \qquad (36)$$

$$\lambda_2[p_2q_2\sin(\vartheta_2-\omega_2) - pq\sin(\vartheta_2-\omega)] + \lambda_3\Big(\frac{\partial\psi}{\partial\vartheta_2} - \frac{\partial\psi_2}{\partial\vartheta_2}\Big) = 0, \qquad (37)$$

$$-p\sqrt{1-\delta}+\frac{p_1^3}{p}+q_1\left(\frac{p_1^2}{p^2}-1\right)\cos(\vartheta_1-\omega_1)+\lambda_1[2p+q\cos(\vartheta_1-\omega)]+$$
$$+\lambda_2[2p+q\cos(\vartheta_2-\omega)]+\lambda_3\frac{\partial\psi}{\partial p}=0, \qquad (38)$$

$$q\sqrt{1-\delta}-q_1\cos(\omega_1-\omega)+\lambda_1 p\cos(\vartheta_1-\omega)+\lambda_2 p\cos(\vartheta_2-\omega)+\lambda_3\frac{\partial\psi}{\partial q}=0, \qquad (39)$$

$$-qq_1\sin(\omega_1-\omega)+\lambda_1 pq\sin(\vartheta_1-\omega)+\lambda_2 pq\sin(\vartheta_2-\omega)+\lambda_3\frac{\partial\psi}{\partial\omega}=0. \qquad (40)$$

The obtained equations (36) – (40), alongside with the equations (27) – (29) form a system of required conditions, consisting of eight equations with eight unknowns : $p, q, \omega, \vartheta_1, \vartheta_2, \lambda_1, \lambda_2, \lambda_3$. These equations ought to be resolved consistently.

Remark 2.- If the problem under consideration does not account for concrete motions, that is, if the initial configuration of heavenly objects on boundary orbits is arbitrary, the condition (29) of motion time coincidence before encounter at the final point should be dropped, and we should postulate $\lambda_3 = 0$ in the remaining system of seven equations.

Remark 3.- The light pressure on the final orbit is easily taken into account. To that effect it is sufficient to estimate the elements p_2, q_2, ω_2, T_2 as geometrical, and to substitute in all equations the function ψ_2 by

$$\widetilde{\psi}_2=\psi_2(1-\delta_2)^{-\frac{1}{2}},$$

where δ_2 is the parameter δ, computed for a heavenly object situated on the final orbit.

4. - FLIGHT ALONG CIRCULAR ORBITS

Let the initial and final orbits be circular, respectively of radii r_1, r_2 and quantities $q_1 = 0$, $q_2 = 0$.

It may be shown that from the system of necessary conditions, it follows

$$\lambda_3=0. \qquad (41)$$

Therefore, we must resolve the problem first without taking into account concrete motions; the solution will contain only the differences $\vartheta_1-\omega$, $\vartheta_2-\omega$ and, consequently, one of the angles $\vartheta_1, \vartheta_2, \omega$ will be arbitrary.

After that, it is easy to find from the condition (29) of motion time coincidence before encounter, for example, the angle ϑ_1

$$\vartheta_1 = [(\alpha-\psi)p_1^3p_2^3 + \omega_1 p_2^3 - \omega_2 p_1^3 + f p_1^3](p_2^3 - p_1^3)^{-1}, \qquad (42)$$

where $\frac{\psi}{K}$ is the motion time along the optimum orbit; $f=\vartheta_2-\vartheta_1$ is the difference in the true anomalies of the starting and finishing points.

Let us consider the problem of flight without taking into account the concrete motions. It follows from equations (36), (37) that

$$\lambda_1 \sin(\vartheta_1-\omega)=0,\ \lambda_2 \sin(\vartheta_2-\omega)=0. \qquad (43)$$

The last equations may be satisfied by three methods, since the conditions $\lambda_1=0$, $\lambda_2=0$, contradict the equation (39).

<u>1st Method</u>.- Let us postulate

$$\sin(\vartheta_1-\omega)=0,\ \sin(\vartheta_2-\omega)=0. \qquad (44)$$

In this case the only solution (with a precision to arbitrary choice of the angle of the start point ϑ_1) is the Homan ellipse in geometrical elements. For it

$$p=\sqrt{\frac{p_1^2+p_2^2}{2}},\quad q=\pm\frac{p_1^2-p_2^2}{\sqrt{2(p_1^2+p_2^2)}}, \qquad (45)$$

$$\lambda_1=\frac{p_2^4\sqrt{1-\delta}-p_1^3 p}{4p^4},\ \lambda_2=\frac{p_1^3(p_1\sqrt{1-\delta}-p)}{4p^4}, \qquad (46)$$

$$\Delta U=\frac{Kp_1}{p}|p_1\sqrt{1-\delta}-p|. \qquad (47)$$

The upper signs respond to the case of flights to orbit of greater radius

$$r_2>r_1,\ p_2<p<p_1,\ \vartheta_1=\omega,\ \vartheta_2=\omega+\pi,\ \Phi=0 \text{ at } p<p_1\sqrt{1-\delta},$$
$$\Phi=\pi \text{ at } p>p_1\sqrt{1-\delta}; \qquad (48)$$

the lower ones — to the case of flights to orbits of lesser radius

$$r_2<r_1,\quad p_2>p>p_1,\quad \vartheta_1=\omega-\pi,\quad \vartheta_2=\omega,\quad \Phi=\pi. \qquad (49)$$

<u>2nd Method</u>.- We shall satisfy the equations (43) in the following manner :

$$\sin(\vartheta_1-\omega)=0,\ \lambda_2=0. \qquad (50)$$

From the remaining equations of the system of necessary conditions we shall find

$$p=p_1\sqrt{1-\delta},\quad q=\frac{p_1\delta}{\sqrt{1-\delta}},\tag{51}$$

$$\vartheta_1=\omega,\quad \cos(\vartheta_2-\omega)=\frac{p_2^2-p_1^2(1-\delta)}{p_1^2\delta},\tag{52}$$

$$\Delta U=0.\tag{53}$$

If $\delta<0.5$, the solution exists only at

$$\frac{r_1}{1-2\delta}>r_2>r_1,\quad \sqrt{1-2\delta}\,p_1<p_2<p_1,\tag{54}$$

but if $\delta>0.5$, it exists only at $r_2>r_1$.

The escape takes place at pericenter of flight orbit; no additional fuel consumption is required, and that is why from the standpoint of energy these orbits are more advantageous than the Homan ellipse. During flights of shell-sondes, it is sufficient to inflate the shell in order to put them in the obtained orbits. Let us remark, that at $r_2=\frac{r_1}{1-2\delta}$ the orbit obtained will coincide with the Homan ellipse; at $\delta=0.5$ it will be a parabola, and at $\delta>0.5$ — a hyperbola, in both of which the Sun will be situated in the inner focus.

3rd Method. - In this case

$$\lambda_1=0,\quad \sin(\vartheta_2-\omega)=0.\tag{55}$$

The remaining equations of the system of necessary conditions will give

$$p=\frac{p_2^4\sqrt{1-\delta}}{p_1^3},\quad q=\frac{p_1^6-p_2^6(1-\delta)}{p_1^3p_2^2\sqrt{1-\delta}},\tag{56}$$

$$\cos(\vartheta_1-\omega)=\frac{p_1^8-p_2^8(1-\delta)}{p_2^2\left[p_1^6-p_2^6(1-\delta)\right]},\quad \vartheta_2=\omega.\tag{57}$$

Such orbits are only possible at

$$\sigma^2r_1\leqslant r_2<r_1,\quad p_2>p_1>\sigma p_2,\tag{58}$$

where σ is the unique positive root of the equation

$$\sigma^8+\sigma^6=2(1-\delta).\tag{59}$$

The finishing point coincides with the flight orbit pericenter. At $r^2=\sigma^2 r$, the orbit obtained coincides with the Homan ellipse. The characteristic velocity and the tangent of thrust's inclination angle will be determined as follows

$$\Delta U = K\left\{3p_1^2 - 2p_2^2 - \frac{p_1^6}{p_2^4} + 2\delta\left(p_2^2 - p_1^2\right)\right\}^{\frac{1}{2}}, \tag{60}$$

$$\operatorname{tg}\Phi = \frac{p_2^2\left[p_1^6 - p_2^6(1-\delta)\right]\sin(\vartheta_1 - \omega)}{p_1^4\left(p_1^4 - p_2^4\right)}. \tag{61}$$

The Homan-type flight (45) – (49) is possible in the same interval (58). Let us compose the difference in the squares of characteristic velocities on the Homan ellipse and on the new orbit (56) – (61); it is

$$\frac{K^2 p_2^4}{p_1^2 + p_2^2}\left[\frac{p_1^3\sqrt{p_1^2 + p_2^2}}{p_2^4} - \sqrt{2(1-\delta)}\right]^2 > 0. \tag{62}$$

Consequently, the orbits obtained are more optimum than the Homan ellipse.

Therefore, the function (34) takes the least value on orbits considered in the methods 2 and 3, provided the conditions (54) or (58) are respectively fulfilled, and on the Homan ellipse (45) – (49) in the opposite case.

Remark 4.- There any many problems where the parameter δ is a small quantity; in this case the approximate value of the quantity σ is

$$\sigma = 1 - \frac{1}{7}\delta - \frac{43}{686}\delta^2 + \ldots . \tag{63}$$

5. - FLIGHT BETWEEN ORBITS OF SMALL ECCENTRICITIES

Assuming that q_1 and q_2 are small, let us introduce the small parameter ε according to formulas

$$q_1 = q_1'\varepsilon, \quad q_2 = q_2'\varepsilon. \tag{64}$$

We shall seek the solutions of the equations (27) – (29), (36) – (40) in the form of series by powers ε ; we shall denote by strokes the coefficients of the series sought for at first powers ε. At $\delta = 0$, analogous expansions were obtained in the work [3].

The solution will be conducted in the assumption that r_1, r_2 do not lie in the regions (54) or (58), that is, we shall take the Homan ellipse for the zero approximation.

Assume now that at first the concrete motions are not taken into account. We have in the approximation of the zero order relative to ε a unique solution — the Homan ellipse (with a precision to arbitrary choice of one of the angles $\vartheta_1, \vartheta_2, \omega$), corresponding to the system (45) — (49). The angular distance of the pericenter ω is determined from the first order approximation with the help of the equations (36), (37) and (40), as follows :

$$\operatorname{tg}\omega = \frac{q_1' p_1\left[\lambda_1 + \frac{2p_1}{p} - \frac{2-\delta}{\sqrt{1-\delta}}\right]\sin\omega_1 - q_2' p_2\lambda_2\sin\omega_2}{q_1' p_1\left[\lambda_1 + \frac{2p_1}{p} - \frac{2-\delta}{\sqrt{1-\delta}}\right]\cos\omega_1 - q_2' p_2\lambda_2\cos\omega_2}. \tag{65}$$

The exact solution of the first order is given by the following formulas:

$$\left.\begin{aligned} p' &= \frac{q_1' p_1\cos(\vartheta_1-\omega_1) + q_2' p_2\cos(\vartheta_2-\omega_2)}{4p}, \\ q' &= -\frac{q_1' p_1(q\mp 2p)\cos(\vartheta_1-\omega_1) + q_2' p_2(q\pm 2p)\cos(\vartheta_2-\omega_2)}{4p^2}, \end{aligned}\right\} \tag{66}$$

$$\omega' = \frac{-\alpha_* A - \beta_* B + C}{\alpha_* + \beta_* + \gamma_*}, \quad \vartheta_1' = \omega' + A, \quad \vartheta_2' = \omega' + B, \tag{67}$$

$$\lambda_1' = \frac{pR_* - (q\mp 2p)S_*}{4p^3}, \quad \lambda_2' = \frac{pR_* - (q\pm 2p)S_*}{4p^3}, \tag{68}$$

$$\begin{aligned} \Delta U' = \frac{K^2\sqrt{1-\delta}}{\Delta U}\Biggl\{ & \frac{p_1^3 - p^3\sqrt{1-\delta}}{p^2}p' + q\sqrt{1-\delta}\,q' + \\ & + \frac{q_1' p_1(2p - 2p_1\sqrt{1-\delta} - p\delta)\cos(\vartheta_1-\omega_1)}{p\sqrt{1-\delta}}\Biggr\}, \end{aligned} \tag{69}$$

$$\Phi' = \pm\frac{pq\sqrt{1-\delta}(\vartheta_1' - \omega')}{p_1(p_1\sqrt{1-\delta}-p)} - \frac{q_1' p\sin(\vartheta_1-\omega_1)}{p_1(p_1\sqrt{1-\delta}-p)}, \tag{70}$$

$$\left.\begin{aligned} A &= \pm\frac{q_1 p_1}{\lambda_1 pq}\left[\frac{p}{p_1} + \frac{p_1}{p} + \lambda_1 - \frac{2-\delta}{\sqrt{1-\delta}}\right]\sin(\vartheta_1-\omega_1), \\ B &= \mp\frac{q_2' p_2\sin(\vartheta_2-\omega_2)}{pq}, \end{aligned}\right\} \tag{71}$$

$$C = q_1'\left(\frac{p_1^2-p^2}{p^2}p' - p_1\lambda_1' \mp q'\right)\sin(\vartheta_1-\omega_1) - q_2'\lambda_2' p_2\sin(\vartheta_2-\omega_2), \tag{72}$$

$$\left.\begin{aligned} \alpha_* &= q_1 p_1\left[\frac{p}{p_1} + \frac{p_1}{p} + \lambda_1 - \frac{2-\delta}{\sqrt{1-\delta}}\right]\cos(\vartheta_1-\omega_1), \\ \beta_* &= q_2' p_2\lambda_2\cos(\vartheta_2-\omega_2), \\ \gamma_* &= \pm q_1' q\cos(\vartheta_1-\omega_1), \end{aligned}\right\} \tag{73}$$

$$R_* = q_1'\left(1 - \frac{p_1^2}{p^2}\right)\cos(\vartheta_1 - \omega_1) +$$
$$+\left(\sqrt{1-\delta} + \frac{2p_1^3}{p^3} - 2\lambda_1 - 2\lambda_2\right)p' \mp (\lambda_1 - \lambda_2)q', \tag{74}$$
$$S_* = \pm q_1' \cos(\vartheta_1 - \omega_1) \mp (\lambda_1 - \lambda_2)p' - \sqrt{1-\delta}q' \tag{75}$$

The solution of the problem of an optimum energy one-impulse flight without taking into account the light pressure (see [3]) with a precision to terms of the first order ε is obtained from the formulas (45) — (49) and (65) — (75), provided we postulate in the latter $\delta = 0$. In the case when the parameter δ is of the same order with eccentricities of boundary orbits

$$\delta = \varepsilon\delta', \tag{76}$$

the first order corrections to this solution, conditioned by light pressure will be

$$\left.\begin{gathered}
p' = 0, \quad q' = 0 \\
\omega' = \vartheta_1' = \vartheta_2' = \frac{2q_1' q_2' p_1^4 p_2 p^3 (p_1 - p)^2 \sin(\omega_2 - \omega_1)\cos^2\omega}{\left\{q_1' p_1\left[8p^3(p_1 - p) + p_2^4 - p_1^3 p\right]\cos\omega_1 - q_2' p_1^3 p_2 (p_1 - p)\cos\omega_2\right\}^2}\delta'. \\
\lambda_1' = -\frac{p_2^4}{8p^4}\delta', \quad \lambda_2' = -\frac{p_1^4}{8p^4}\delta'. \\
\Delta U' = \mp\frac{p_1^2}{2p}\delta', \quad \Phi' = 0.
\end{gathered}\right\} \tag{77}$$

Thus, the accounting of the effect of light pressure at a small parameter δ and boundary orbits of small eccentricities has been reduced to rotating the flight orbit by a magnitude of the first order relative to the orbit corresponding to the solution of the optimum problem without taking into account the light pressure. The dimension and shape of the orbit and the true anomalies of the starting and finishing points are varying over it by magnitudes of the second order only.

Let us pass now to the solution of the problem, taking into account the concrete motions. In the zero order approximation we have a unique solution — the Homan ellipse (45) — (49), with equality (41) fulfilled, and for the function ψ we have :

$$\psi = \frac{\pi}{(p^2 - q^3)^{\frac{3}{2}} \sqrt{1-\delta}}. \tag{78}$$

The starting angle ϑ_1 is found from the equality (42) as follows :

$$\vartheta_1 = \pi \frac{\left(\frac{p_1^2 + p_2^2}{2}\right)^{\frac{3}{2}} \frac{1}{\sqrt{1-\delta}} - p_1^3}{p_1^3 - p_2^3} + \frac{\omega_2 p_1^3 - \omega_1 p_2^3 - \alpha p_1^3 p_2^3}{p_1^3 - p_2^3}. \tag{79}$$

In the first order approximation we shall find from the system of necessary conditions for p', q', the very same formulas (66), and for ϑ_1', ϑ_2', ω' we shall have

$$\vartheta_1' = \frac{\widetilde{A}\left(p_1^4 p_2 \sqrt{1-\delta} - p p_2^4\right) - \widetilde{B}\left(p_1^4 p_2 \sqrt{1-\delta} - p p_1^4\right) - \widetilde{C} p_1^4 p_2^4 \sqrt{1-\delta}}{p_1 p_2 \left(p_1^3 - p_2^3\right)\sqrt{1-\delta}}, \tag{80}$$

$$\vartheta_2' = \frac{\widetilde{A}\left(p_2^4 p_1 \sqrt{1-\delta} - p p_2^4\right) - \widetilde{B}\left(p_2^4 p_1 \sqrt{1-\delta} - p p_1^4\right) - \widetilde{C} p_1^4 p_2^4 \sqrt{1-\delta}}{p_1 p_2 \left(p_1^3 - p_2^3\right)\sqrt{1-\delta}}, \tag{81}$$

$$\omega' = \frac{\widetilde{A}\left(p_2^4 p_1 \sqrt{1-\delta} - p p_2^4\right) - \widetilde{B}\left(p_1^4 p_2 \sqrt{1-\delta} - p p_1^4\right) - \widetilde{C} p_1^4 p_2^4 \sqrt{1-\delta}}{p_1 p_2 \left(p_1^3 - p_2^3\right)\sqrt{1-\delta}}, \tag{82}$$

where

$$\widetilde{A} = A \pm \frac{p_1\sqrt{1-\delta} - p}{\lambda_1 p_1^4 p q \sqrt{1-\delta}} \lambda_3', \quad \widetilde{B} = B \mp \frac{p - p_2\sqrt{1-\delta}}{\lambda_2 p_2^4 p q \sqrt{1-\delta}} \lambda_3', \tag{83}$$

$$\widetilde{C} = \frac{3\pi (pp' - qq')}{(p^2 - q^2)^{\frac{5}{2}} \sqrt{1-\delta}} + \frac{2q_1' \sin(\vartheta_1 - \omega_1)}{p_1^4} - \frac{2q_2' \sin(\vartheta_2 - \omega_2)}{p_2^4}. \tag{84}$$

The functions A and B are computed by the formulas (71), and

$$\lambda_3' = \frac{q_1' p_1 \left(\frac{2p_1}{p} + \lambda_1 - \frac{2-\delta}{\sqrt{1-\delta}}\right) \sin(\vartheta_1 - \omega_1) + q_2' p_2 \lambda_2 \sin(\vartheta_2 - \omega_2)}{p_2^{-3} - p_1^{-3}}. \tag{85}$$

6.- CONDITION AT WHICH THE UTILIZATION OF HOLLOW SHELL-SONDES FOR THE TRANSFER OF USEFUL PAYLOADS IS PROFITABLE FROM THE STANDPOINT OF ENERGY

Let us examine the question relative to the case, when it is more advantageous from the viewpoint of fuel consumption to put a payload of mass m_0 to final orbit with the aid of a shell-sonde, by comparison with its direct placing to the corresponding flight orbit.

In the last case we may neglect the light pressure. We shall estimate that one and the same characteristic velocity will be required in both cases in order to o ercome the attraction of the heavenly body, from which, or from whose satellite, the start takes place. We shall denote this characteristic velocity by $\widetilde{U}$. We shall limit ourselves to the case of circular boundary orbits, with $r_2 > r_1$. At $r_1 > r_2$ it is more advantageous to dispatch the payload in such a fashion, that the resultant of light repulsion forces be as small as possible.

At flight without taking into account the light pressure the Homan ellipse gives fuel consumption its minimum. The characteristic velocity of the initial impulse ΔU_Γ is determined as follows :

$$\Delta U_\Gamma = \frac{K p_1 (p_1 - p)}{p}. \tag{86}$$

The minimum value of the characteristic velocity at flight, taking into account the influence of light pressure $(\delta \leq 0.5)$ is

$$\Delta U = 0 \quad \text{at} \quad \frac{r_1}{1-2\delta} > r_2 > r_1, \tag{87}$$

$$\Delta U = \frac{K p_1 (p_1 \sqrt{1-\delta} - p)}{p} \qquad r_2 > \frac{r_1}{1-2\delta}. \tag{88}$$

Let c_1 be the outflow velocity of gases; then, as is well known,

$$\Delta U + \widetilde{U} = c_1 \ln \frac{m_H}{m_0 + m_1}, \quad \Delta U_\Gamma + \widetilde{U} = c_1 \ln \frac{m_H - m_1 + \Delta m}{m_0}, \tag{89}$$

where m_x is the initial mass of the cosmic device at placing the payload together with the shell-sonde; Δm is the fuel mass variation at direct placing of the payload into orbit; m_1 is the mass of shell-sonde, for which the following is valid :

$$m_1 = 4\pi R^2 \gamma h. \tag{90}$$

At $\Delta m > 0$, it is more advantageous to use a shell-sonde; at $\Delta m < 0$ the lesser fuel consumption will take place at direct placing of the payload into the orbit of the flight.

From the correlations (89) it is easy to find

$$\Delta m = m_0 \left(e^{\frac{\Delta U_\Gamma + \widetilde{U}}{c_1}} - e^{\frac{\Delta U + \widetilde{U}}{c_1}} \right) - m_1 \left(e^{\frac{\Delta U + \widetilde{U}}{c_1}} - 1 \right), \tag{91}$$

where $\Delta U_\Gamma, \Delta U$ are determined from (86)-(89). The last correlation allows at concrete values of $\gamma, h, R, m_0, p_2, p_1, \widetilde{U}$ to estimate, which method of payload delivery is preferable from the standpoint of energy.

The condition of lesser fuel expenditure at utilization of the shell-sonde is obtained in the form

$$\chi = \frac{m_0}{m_1} > \frac{e^{\frac{\Delta U}{c_1}} - e^{-\frac{\widetilde{U}}{c_1}}}{e^{\frac{\Delta U_\Gamma}{c_1}} - e^{\frac{\Delta U}{c_1}}}, \tag{92}$$

It should, however, be borne in mind that the right-hand part depends on the parameter δ, which, in its turn, depends on χ as follows:

$$\delta = \frac{E_\delta r_\delta^2}{4K^2\gamma h(1+\chi)c}. \tag{93}$$

Formulas (92) and (93) allow to determine the dependence between γ, h, χ, p_1, p_2, $\bar{U}$ sought for.

In conclusion, I avail myself of the opportunity to express my sincere gratitude to my scientific guide, Prof. V.S.Novoselov, for his constant help in the course of the work.

*** THE END ***

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